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The range of united K-theory

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Abstract

We prove that the united K-theory functor is a surjective functor from the category of real simple separable purely infinite C^*-algebras to the category of countable acyclic *CRT*-modules. As a consequence, we show that every complex Kirchberg algebra satisfying the universal coefficient theorem is the complexification of a real C^*-algebra.

Keywords: Real C^*-algebra; K-theory

1. Introduction

In this paper we further investigate the united K-theory functor for real C^*-algebras, developed in [2]. We will show in Theorem 1 that united K-theory is a surjective functor from the category of real simple purely infinite C^*-algebras to the category of countable acyclic *CRT*-modules. Thus, an example of a real separable Kirchberg algebra can be produced simply by specifying the prescribed united K-theory. Furthermore, we may require that its complexification is in Schochet's bootstrap category $\mathcal{C}$ (see [21] or [20]) and, subject to this constraint, the real C^*-algebra obtained is unique up to KK-equivalence, by the Universal Coefficient theorem for real C^*-algebras in [3]. In particular, given a complex Kirchberg algebra A in the bootstrap category, the KK-equivalence classes of real structures on A are in one-to-one correspondence with the isomorphism classes of *CRT*-modules whose complex part is isomorphic to the K-theory of A.

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Our conjecture is that there is an analog for real C^*-algebras of the classification theorem of Kirchberg [13] and Phillips [15] stating that two real Kirchberg algebras are *KK*-equivalent if and only they are isomorphic. The universal coefficient theorem of [3] and the surjectivity result of the present paper give us confidence in this conjecture and that united K-theory is the right invariant to use in this context.

Indeed, neither real K-theory nor complex K-theory (that is, the K-theory of the complexification) by itself can do the job of united K-theory. In [2], we showed that the two tensor products of real Cuntz algebras $\mathcal{O}_3^{\mathbb{R}} \otimes \mathcal{O}_3^{\mathbb{R}}$ and $\mathcal{O}_3^{\mathbb{R}} \otimes \mathcal{O}_5^{\mathbb{R}}$ are non-isomorphic although their complexifications are isomorphic. Many more such examples can be obtained by applying Theorem 1 (using, for example, the *CRT*-modules $M_i = \Sigma^i K^{CRT}(\mathbb{R})$ for $i = 0, 2, 4, 6$). Hence complex K-theory by itself is not sufficient to classify real simple purely infinite C^*-algebras. Neither is real K-theory by itself sufficient, as shown by Theorem 16, also a corollary of Theorem 1. (However, results of [12] indicate that it may very well be that the real and complex parts together are enough.)

We will further apply our main results to the question of determining which C^*-algebras are complexifications of real C^*-algebras. In [8,16,17], examples of C^*-algebras not isomorphic to their own opposite algebra are described. It follows that these C^*-algebra are not isomorphic to the complexification of any real C^*-algebra since $\mathbb{C} \otimes A$ has an anti-multiplicative automorphism $a \mapsto \bar{a}^*$ for any real C^*-algebra A. On the other hand, the Kirchberg–Phillips classification theorem implies that any complex Kirchberg algebra satisfying the universal coefficient theorem (UCT) is isomorphic to its opposite algebra (since they have the same K-theory). In the present paper (in Theorem 17) we will show that any such algebra is in fact the complexification of a real C^*-algebra.

The main theorem is stated in the next section following a short review of united K-theory and *CRT*-modules. The proof of the main theorem takes place through a series of approximating steps. In Section 3, we first show how to obtain a real separable C^*-algebra whose united K-theory is isomorphic to the prescribed *CRT*-module. In Section 4 we show how to modify this algebra to form a real unital C^*-algebra with the same K-theory. Finally in Section 5 we use a real version of Kumjian's construction in [14] to obtain an algebra which is simple and purely infinite, completing the proof of the main theorem.

In Section 6, we will combine our main theorem with *CRT*-module constructions of [12] to obtain two results already mentioned: that two non-isomorphic real Kirchberg C^*-algebras can have the same real K-theory and that every complex Kirchberg algebra satisfying the UCT has a real structure.

2. United *K*-theory

Recall that for a real C^*-algebra A, the united K-theory $K^{CRT}(A)$ defined in [2] consists of three graded modules and the collection of natural transformations between them. The three objects are:

(1) real K-theory $KO_*(A)$—defined to be the K-theory of the real C^*-algebra A as discussed, for example, in [23];
(2) complex K-theory $KU_*(A)$—defined to be the K-theory of the complexification $A_{\mathbb{C}} = \mathbb{C} \otimes A$;
(3) self-conjugate K-theory $KT_*(A)$—defined to be the K-theory of $T \otimes A = \{f : [0,1] \to \mathbb{C} \otimes A \mid f(0) = \overline{f(1)}\}$.

These objects are taken not just as graded groups, but as graded modules over the graded unital rings $KO_*(\mathbb{R})$, $KU_*(\mathbb{R})$, and $KT_*(\mathbb{R})$, respectively; which are displayed here in degrees 0–8:

$$\begin{array}{l} K_*(\mathbb{R}) = \mathbb{Z}\ \mathbb{Z}_2\ \mathbb{Z}_2\ 0\ \mathbb{Z}\ 0\ 0\ 0\ \mathbb{Z}, \\ K_*(\mathbb{C}) = \mathbb{Z}\ 0\ \mathbb{Z}\ 0\ \mathbb{Z}\ 0\ \mathbb{Z}\ 0\ \mathbb{Z}, \\ K_*(T) = \mathbb{Z}\ \mathbb{Z}_2\ 0\ \mathbb{Z}\ \mathbb{Z}\ \mathbb{Z}_2\ 0\ \mathbb{Z}\ \mathbb{Z} \end{array}$$

(see [23, p. 23] and [2, Tables 1–3]). The generators are the elements $1_O \in K_0(\mathbb{R})$, $\eta_O \in K_1(\mathbb{R})$, $\eta_O^2 \in K_2(\mathbb{R})$, $\xi \in K_4(\mathbb{R})$, and the invertible element $\beta_O \in K_8(\mathbb{R})$. The ring $K_*(\mathbb{C})$ is the free unital polynomial ring generated by the invertible Bott element $\beta_U \in K_2(\mathbb{C})$. The ring $K_*(T)$ has generators 1_T in degree 0, η_T in degree 1, ω in degree 3, and the invertible element β_T in degree 4. Thus, $KO_*(A)$ has period 8, $KU_*(A)$ has period 2, and $KT_*(A)$ has period 4. The natural transformations among these graded groups are

$$\begin{aligned} c_n &: KO_n(A) \longrightarrow KU_n(A), & r_n &: KU_n(A) \longrightarrow KO_n(A), \\ \varepsilon_n &: KO_n(A) \longrightarrow KT_n(A), & \zeta_n &: KT_n(A) \longrightarrow KU_n(A), \\ (\psi_U)_n &: KU_n(A) \longrightarrow KU_n(A), & (\psi_T)_n &: KT_n(A) \longrightarrow KT_n(A), \\ \gamma_n &: KU_n(A) \longrightarrow KT_{n-1}(A), & \tau_n &: KT_n(A) \longrightarrow KO_{n+1}(A), \end{aligned}$$

where, for example, the complexification operation c is induced by the inclusion $A \to \mathbb{C} \otimes A$ and the realification operation r is induced by the inclusion $\mathbb{C} \otimes A \to M_2(\mathbb{R}) \otimes A$. For descriptions of the other operations, see [2, Sections 1.1 and 1.2].

The target category of united K-theory is the category of abstract CRT-modules described in [5]. An abstract CRT-module is a triple $M = (M^O, M^U, M^T)$ consisting of graded modules over $KO_*(\mathbb{R})$, $KU_*(\mathbb{R})$, and $KT_*(\mathbb{R})$, respectively. Furthermore there must be $KO_*(\mathbb{R})$-module homomorphisms r, c, ε, ζ, ψ_U, ψ_T, γ and τ which satisfy the relations

$$\begin{aligned} rc &= 2, & \psi_U\beta_U &= -\beta_U\psi_U, & \xi &= r\beta_U^2 c, \\ cr &= 1 + \psi_U, & \psi_T\beta_T &= \beta_T\psi_T, & \omega &= \beta_T\gamma\zeta, \\ r &= \tau\gamma, & \varepsilon\beta_O &= \beta_T^2\varepsilon, & \beta_T\varepsilon\tau &= \varepsilon\tau\beta_T + \eta_T\beta_T, \\ c &= \zeta\varepsilon, & \zeta\beta_T &= \beta_U^2\zeta, & \varepsilon r\zeta &= 1 + \psi_T, \\ (\psi_U)^2 &= 1, & \gamma\beta_U^2 &= \beta_T\gamma, & \gamma c\tau &= 1 - \psi_T, \\ (\psi_T)^2 &= 1, & \tau\beta_T^2 &= \beta_O\tau, & \tau &= -\tau\psi_T, \\ \psi_T\varepsilon &= \varepsilon, & \gamma &= \gamma\psi_U, & \tau\beta_T\varepsilon &= 0, \\ \zeta\gamma &= 0, & \eta_O &= \tau\varepsilon, & \varepsilon\xi &= 2\beta_T\varepsilon, \\ \zeta &= \psi_U\zeta, & \eta_T &= \gamma\beta_U\zeta, & \xi\tau &= 2\tau\beta_T \end{aligned}$$

as in [5, Section 1.9]. These relations are satisfied by united K-theory by [2, Proposition 1.7].

Not every abstract CRT-module M can be realized as the united K-theory of a real C^*-algebra. According to [2, Theorem 1.18] a necessary condition is that M be acyclic, i.e., the following complexes must be exact:

$$\cdots \to M_n^O \xrightarrow{\eta_O} M_{n+1}^O \xrightarrow{c} M_{n+1}^U \xrightarrow{r\beta_U^{-1}} M_{n-1}^O \to \cdots,$$

$$\cdots \to M_n^O \xrightarrow{\eta_O^2} M_{n+2}^O \xrightarrow{\varepsilon} M_{n+2}^T \xrightarrow{\tau\beta_T^{-1}} M_{n-1}^O \to \cdots,$$

$$\cdots \to M_{n+1}^U \xrightarrow{\gamma} M_n^T \xrightarrow{\zeta} M_n^U \xrightarrow{1-\psi_U} M_n^U \to \cdots.$$

Our main theorem is that every countable acyclic *CRT*-module can be realized as the united K-theory of a real C^*-algebra. Furthermore, the real C^*-algebra can be taken to be simple and purely infinite. Following [19], we say that a complex C^*-algebra is a Kirchberg algebra if it is separable, nuclear, simple, and purely infinite. We say that a real C^*-algebra A is a Kirchberg algebra if the complexification $A_{\mathbb{C}}$ is a Kirchberg algebra; this implies that A is also simple and purely infinite.

Indeed, any real C^*-algebra is simple if its complexification is simple; for if I is a closed ideal in A, then $I_{\mathbb{C}}$ is a closed ideal in $A_{\mathbb{C}}$. The converse is not true in general. In fact, the complexification $A_{\mathbb{C}}$ is simple if and only if A is simple and is not itself isomorphic to a complex C^*-algebra.

Following the definition in [24], a real C^*-algebra A is purely infinite if each hereditary subalgebra of the form $\overline{xAx}$ for a nonzero positive element x contains an infinite projection. Theorem 3.3 of [24] (the proof of which was corrected in [4]) states that A is purely infinite if $A_{\mathbb{C}}$ is purely infinite. The converse is still an open question, although there is a partial result in [24, Section 4].

Theorem 1.

(1) *Let M be any countable acyclic CRT-module. Then there exists a real stable Kirchberg algebra A such that $K^{CRT}(A) \cong M$ and $A_{\mathbb{C}}$ satisfies the UCT.*
(2) *Let M be any countable acyclic CRT-module and let m be any element of M_0^O (that is, m is a degree zero element in the real part of M). Then there exists a real unital Kirchberg algebra A such that $(K^{CRT}(A), [1_A]) \cong (M, m)$ and $A_{\mathbb{C}}$ satisfies the UCT.*

In [3], we developed united *KK*-theory and proved a Universal Coefficient theorem for real C^*-algebras. This UCT implies that two separable C^*-algebras A and B such that $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$ are in the bootstrap category are *KK*-equivalent (in the real sense) if and only if $K^{CRT}(A)$ and $K^{CRT}(B)$ are isomorphic *CRT*-modules. Therefore, we have the following immediate corollaries.

Corollary 2.

(1) *There is an equivalence between the category of KK-equivalence classes of real stable separable C^*-algebras satisfying the UCT and the category of all countable acyclic CRT-modules.*
(2) *There is an equivalence between the category of KK-equivalence classes of real unital separable C^*-algebras satisfying the UCT and the category of all countable acyclic CRT-modules M with specified element $m \in M_0^O$.*

Corollary 3.

(1) *Let B be a complex stable Kirchberg algebra satisfying the UCT. Then there is a bijective correspondence between isomorphism classes of real C^*-algebras A such that $A_{\mathbb{C}} \cong B$ and isomorphism classes of CRT-modules M such that $M^U \cong K_*(B)$.*
(2) *Let B be a complex unital Kirchberg algebra satisfying the UCT. Then there is a bijective correspondence between isomorphism classes of real C^*-algebras A such that $A_{\mathbb{C}} \cong B$ and isomorphism classes of pairs (M, m) where M is a CRT-module such that $m \in M_0^O$ and $M^U \cong K_*(B)$.*

Finally, we state here for the record the following important theorem which will be used frequently in the sequel. It is implicit in [2], being an immediate consequence of the results of [5, Section 2.3] (restated as [2, Propositions 1.14 and 1.15]) and [2, Theorem 1.12].

Theorem 4.

(1) *Let A be a real C^*-algebra. If one of the three graded modules $KO_*(A)$, $KU_*(A)$, and $KT_*(A)$ is trivial, then all three are trivial.*
(2) *Let $f : A \to B$ be a homomorphism of real C^*-algebras. If one of the three graded homomorphisms $f_* : KO_*(A) \to KO_*(B)$, $f_* : KU_*(A) \to KU_*(B)$, and $f_* : KT_*(A) \to KT_*(B)$ is an isomorphism, then all three are isomorphisms.*

3. The first C^*-algebra construction

For any acyclic CRT-module M, there is, according to [6, Theorem 2.9], a topological spectrum E such that $K^{CRT}(E) \cong M$. It is not known in general whether E can be taken to be a actual topological space; however, by [5, Theorem 11.1], it is possible to find a CW-complex X such that $K^{CRT}(X) \cong M$ if M is finitely generated. In this section, we prove the following theorem which only requires that M be countable, but leaves the commutative setting far behind.

Theorem 5. *Let M be a countable acyclic CRT-module. Then there is a real separable nuclear C^*-algebra A satisfying the UCT such that $K^{CRT}(A) \cong M$.*

First we establish some preliminary notation. Given a real C^*-algebra A we define the suspension by $SA = C_0(\mathbb{R}, A)$ and the desuspension by

$$S^{-1}A = \{ f \in C_0(\mathbb{R}, \mathbb{C} \otimes A) \mid f(-x) = \overline{f(x)} \}.$$

This nomenclature is justified by the result that $SS^{-1}\mathbb{R}$ and $S^{-1}S\mathbb{R}$ are KK-equivalent to $\mathbb{R}$ [2, Proposition 1.20]. More generally, we define

$$S^n A = \begin{cases} \underbrace{SS\dots S}_{n} A & \text{if } n \geqslant 0, \\ \underbrace{S^{-1}S^{-1}\dots S^{-1}}_{-n} A & \text{if } n < 0. \end{cases}$$

Let ι represent the orientation-reversing involution of SA, which induces multiplication by -1 on K-theory.

Recall from [2, Section 2.1] that $K^{CRT}(\mathbb{R})$, $K^{CRT}(\mathbb{C})$, and $K^{CRT}(T)$ are free CRT-modules. The CRT-module $K^{CRT}(\mathbb{R})$ is generated by 1_O, the class of the identity in $KO_0(\mathbb{R})$; the element $\kappa_1 \in KU_0(\mathbb{C})$ generates $K^{CRT}(\mathbb{C})$ as a CRT-module and satisfies $r(\kappa_1) = 1_U \in KO_0(\mathbb{C})$; and the element $\chi \in KT_{-1}(T)$ generates $K^{CRT}(T)$ and satisfies $\tau(\chi) = 1_T \in KO_0(T)$.

Lemma 6. *Let B be any real unital C^*-algebra and let $x \in KO_0(B)$. Then there is a positive integer n and a C^*-algebra homomorphism $\alpha : S\mathbb{R} \to M_n SB$ such that $\alpha_*(1_O) = x$.*

Note that we are making use of the identifications $KO_0(\mathbb{R}) = KO_{-1}(S\mathbb{R})$ and $KO_0(B) = KO_{-1}(SB)$, claiming that $\alpha_* : KO_{-1}(S\mathbb{R}) \to KO_{-1}(SB)$ sends 1_O to x.

Proof. Let $x = [p_1] - [p_2]$ where p_i is a projection in $M_{n_i}(B)$ for $i = 1, 2$. First define $\alpha_i : \mathbb{R} \to M_{n_i} B$ by $\alpha_i(t) = tp_i$. Then let $n = n_1 + n_2$ and define $\alpha = S\alpha_1 \oplus (S\alpha_2 \circ \iota)$ from $\mathbb{R}$ to $M_n B$. Then $\alpha_*(1_O) = [p_1] + \iota_*[p_2] = [p_1] - [p_2] = x$. $\square$

Lemma 7. *Let B be any real unital C^*-algebra and let $y \in KU_0(B)$. Then there is a positive integer n and a C^*-algebra homomorphism $\alpha : S\mathbb{C} \to M_n S^{-1} B$ such that $\alpha_*(\kappa_1) = \beta_U^{-1} y$.*

Proof. Consider the unital inclusion $c : \mathbb{R} \to \mathbb{C}$. We apply the mapping cone construction to obtain a C^*-algebra homomorphism $S\mathbb{C} \to Cc$. In the proof of [2, Theorem 1.18], we found that the mapping cone Cc is homotopy equivalent to S^{-1}. Let ν be the associated homomorphism $\nu : S\mathbb{C} \to S^{-1}$. Also in [2], it is proven that the element of $KK_{-2}(\mathbb{C}, \mathbb{R})$ represented by ν is $\pm r\beta_U^{-1}$. If the sign is negative, replace ν by $\nu \circ \iota$ to make it positive.

Let $y = [p_1] - [p_2]$ where each p_i is a projection in $M_{n_i}\mathbb{C} \otimes B$. Define a C^*-algebra homomorphism $\rho_i : \mathbb{C} \to M_{n_i}\mathbb{C} \otimes B$ by $\rho_i(t) = tp_i$ for all $t \in \mathbb{C}$. The composition $h_i = \nu \circ S\rho_i$ is a homomorphism from $S\mathbb{C}$ to $M_{n_i} S^{-1} B$ which satisfies $(h_i)_*(1_U) = r\beta_U^{-1}[p_i]$. Let $n = n_1 + n_2$ and define $h = h_1 \oplus (h_2 \circ \iota)$ from $S\mathbb{C}$ to $M_n S^{-1} B$, so that $h_*(1_U) = r\beta_U^{-1}(y)$. Then $rh_*(\kappa_1) = h_* r(\kappa_1) = h_*(1_U) = r\beta_U^{-1}(y)$. Since $\ker r = \operatorname{image} \beta_U^{-1} c$ (by acyclicity [2, Theorem 1.18]), there is an element $x \in KO_1(S^{-1}B) = KO_0(B)$ such that $h_*(\kappa_1) = \beta_U^{-1}(y) + \beta_U^{-1} c(x)$.

To correct the error, let $x = [q_1] - [q_2]$ where q_i is a projection in $M_{m_i} B$ for $i = 1, 2$. Define $\mu_i : \mathbb{R} \to M_{m_i} B$ by $\mu_i(t) = tq_i$ and then define $j_i = S^{-1}\mu_i \circ \nu$. Let $m = m_1 + m_2$ and define $j : S\mathbb{C} \to M_m S^{-1} B$ by $j = (j_1 \circ \iota) \oplus j_2$. Since $\beta_U \nu_*(\kappa_1) = \nu_*(\beta_U \kappa_1) = r\beta_U^{-1} \beta_U \kappa_1 = r\kappa_1 = 1_U$, we have $\nu_*(\kappa_1) = \beta_U^{-1} c(1_O) \in KU_{-2}(\mathbb{R})$. Thus

$$j_*(\kappa_1) = \big((\mu_2)_* - (\mu_1)_*\big)\beta_U^{-1} c(1_O) = -\beta_U^{-1} c\big((\mu_1)_* - (\mu_2)_*\big)(1_O) = -\beta_U^{-1} c(x).$$

We patch together these two homomorphisms by letting $l = m + n$ and defining $\alpha = h \oplus j$ from $S\mathbb{C}$ to $M_l S^{-1} B$. Then $\alpha(\kappa_1) = \beta_U^{-1} y$. $\square$

Lemma 8. *Let B be any real unital C^*-algebra and let $z \in KT_0(B)$. Then there is a positive integer n and a C^*-algebra homomorphism $\alpha : ST \to M_n S^{-2} B$ such that $\alpha_*(\chi) = \beta_T^{-1} z$.*

Proof. The mapping cone of the unital inclusion $\varepsilon : \mathbb{R} \to T$ is homotopy equivalent to S^{-2} (as in the proof of [2, Theorem 1.18]). Thus we obtain a C^*-algebra homomorphism $\sigma : ST \to S^{-2}$.

Also from the proof of Theorem 1.18, the element of $KK_{-3}(T,\mathbb{R})$ represented by σ may be taken to be $\pm\tau\beta_T^{-1}$.

Let $z=[p_1]-[p_2]$ where p_i is a projection in $M_{n_i}T\otimes B$ for $i=1,2$. Since T is commutative, there is a C^*-algebra homomorphism $\rho_i : T\to M_{n_i}T\otimes B$ defined by $\rho_i(t)=tp_i$ for all $t\in T$. The composition $h_i=\sigma\circ S\rho_i$ defines a homomorphism from ST to $M_{n_i}S^{-2}B$ that satisfies $(h_i)_*(1_T)=\tau\beta_T^{-1}[p_i]$. Let $n=n_1+n_2$ and define $h=h_1\oplus(h_2\circ\iota)$ from ST to $M_nS^{-2}B$, so that $h_*(1_T)=\tau\beta_T^{-1}(z)$. Then $\tau h_*(\chi)=h_*\tau(\chi)=h_*(1_T)=\tau\beta_T^{-1}(z)$. Since $\ker\tau=\operatorname{image}\beta_T^{-1}\varepsilon$, there is an element $x\in KO_2(S^{-2}B)=KO_0(B)$ such that $h_*(\chi)=\beta_T^{-1}(z)+\beta_T^{-1}\varepsilon(x)$.

To correct the error, let $x=[q_1]-[q_2]$ where q_i is a projection in $M_{m_i}B$ for $i=1,2$. Define $\mu_i:\mathbb{R}\to M_{m_i}B$ by $\mu_i(t)=tq_i$ and then define $j_i=S^{-2}\mu_i\circ\sigma$. Let $m=m_1+m_2$ and define $j:ST\to M_mS^{-2}B$ by $j=(j_1\circ\iota)\oplus j_2$. Since $\beta_T\sigma_*(\chi)=\sigma_*(\beta_T\chi)=\tau(\chi)=1_T$, we have $\sigma_*(\chi)=\beta_T^{-1}(1_T)$ in $KT_{-4}(\mathbb{R})$. Thus

$$j_*(\chi)=\big((\mu_2)_*-(\mu_1)_*\big)\beta_T^{-1}\varepsilon(1_O)=-\beta_T^{-1}\varepsilon\big((\mu_1)_*-(\mu_2)_*\big)(1_O)=-\beta_T^{-1}\varepsilon(x).$$

We patch these two homomorphisms together by letting $l=m+n$ and defining $\alpha=h\oplus j$ from ST to $M_lS^{-2}B$. $\square$

Proof of Theorem 5. If M is a free CRT-module, then it can be written as a direct sum of monogenic free CRT-modules, and each monogenic CRT-module can be realized as the united K-theory of $\mathbb{R}$, $\mathbb{C}$, T, or a suspension thereof. Therefore, M can be realized as the united K-theory of a direct sum of countably many such C^*-algebras.

Now, let M be an arbitrary countable acyclic CRT-module. By [5, Theorems 3.2 and 3.4], we can find a resolution

$$0\to F_1\xrightarrow{\mu_1}F_0\xrightarrow{\mu_0}M\to 0,$$

where F_0 and F_1 are countable and free CRT-modules.

As in the first paragraph, find real separable C^*-algebras B and C such that $F_0=K^{CRT}(B)$ and $F_1=K^{CRT}(C)$. In particular, we set

$$B=\bigoplus_{i\in I_O}S^{k_i}\mathbb{R}\oplus\bigoplus_{i\in I_U}S^{k_i}\mathbb{C}\oplus\bigoplus_{i\in I_T}S^{k_i}T,$$

where I_O, I_U, and I_T are disjoint countable index sets and where $k_i\in\{0,1,\dots,7\}$ for each i. Our strategy is to realize μ_1 geometrically. That is, we wish to produce a C^*-algebra homomorphism $\beta:B\to C$ whose induced homomorphism on united K-theory is μ_1. Actually, we will replace B and C with algebras B' and C' and the induced homomorphism $\beta_*:K^{CRT}(B')\to K^{CRT}(C')$ will not be identical to μ_1 but will be injective and will have the same cokernel as μ_1.

For any unital C^*-algebra D, let $S^{\sim-}D=(S^{-1}D)^\sim$ denote the unitized desuspension of D and let $S^{\sim-n}D$ denote the n-fold unitized desuspension. Let $C''=S^{10}S^{-2}S^{\sim-8}(C^\sim)$ and let $C'=\mathcal{K}\otimes C''$, where $\mathcal{K}$ is an algebra of compact operators on a separable Hilbert space. Note that $K^{CRT}(C')=K^{CRT}(C)\oplus K^{CRT}(S^{10}S^{-2}S^{\sim-8}\mathbb{R})$ because of the split exact sequence

$$0\to S^{10}S^{-2}S^{\sim-8}\mathbb{R}\to S^{10}S^{-2}S^{\sim-8}C^\sim\to S^{10}S^{-10}C\to 0.$$

For each $i \in I_O$, we construct a geometric realization of the restricted homomorphism

$$K^{CRT}\big(S^{k_i}\mathbb{R}\big) \to K^{CRT}(C)$$

as follows. Let $x \in KO_{-k_i}(C) = KO_0(S^{-k_i}C)$ be the image of $1_O \in KO_{-k_i}(S^{k_i}\mathbb{R}) = KO_0(\mathbb{R})$. By Lemma 6 there is a homomorphism $\alpha_i : S\mathbb{R} \to M_{n_i}S(S^{-k_i}C)^{\sim}$ such that $(\alpha_i)_*(1_O) = x$. Then apply the suspension and desuspension operations to α_i and compose it with the inclusion into C'' to form the homomorphism

$$\beta_i : S^{10}S^{k_i-10}\mathbb{R} \to M_{n_i}S^{10}S^{k_i-10}\big(S^{-k_i}C\big)^{\sim} \hookrightarrow M_{n_i}C''$$

which agrees on united K-theory with the restriction of μ_1 to $K^{CRT}(S^{k_i}\mathbb{R})$.

Similarly, for each $i \in I_U$, consider the restriction of μ_1

$$K^{CRT}\big(S^{k_i}\mathbb{C}\big) \to K^{CRT}(C),$$

and let $y_i \in KU_{-k}(C)$ be the image of $\kappa_1 \in KU_{-k_i}(S^{k_i}\mathbb{C})$. Using Lemma 7, let $\alpha_i : S\mathbb{C} \to M_{n_i}S^{-1}(S^{-k_i}C)^{\sim}$ be given satisfying $(\alpha_i)_*(\kappa_1) = \beta_U^{-1}y$. Again suspend and desuspend to form the composition

$$\beta_i : S^{11}S^{k_i-9}\mathbb{C} \to M_{n_i}S^{10}S^{k_i-10}\big(S^{-k_i}C\big)^{\sim} \hookrightarrow M_{n_i}C''.$$

The induced homomorphism $(\beta_i)_*$ on united K-theory agrees with the restriction of μ_1 to $K^{CRT}(S^{k_i}\mathbb{C})$ up to multiplication by β_U^{-1}. This is not a problem for us; since β_U^{-1} is an isomorphism on united K-theory, the homomorphism $(\beta_i)_*$ is still injective and its image is the same as that of μ_1.

Thirdly, for each $i \in I_U$, consider the restriction of μ_1

$$K^{CRT}\big(S^{k_i}T\big) \to K^{CRT}(C)$$

and let $z_i \in KT_{-k_i-1}(C)$ be the image of $\chi \in KT_{-k_i-1}(S^{k_i}\mathbb{C})$. By Lemma 8, let $\alpha_i : ST \to M_{n_i}S^{-2}(S^{-k_i-1}C)^{\sim}$ be given satisfying $(\alpha_i)_*(\chi) = \beta_T^{-1}z$. Again suspend and desuspend to form

$$\beta_i : S^{11}S^{k_i-7}T \to M_{n_i}S^{10}S^{k_i-9}\big(S^{-k_i-1}C\big)^{\sim} \hookrightarrow M_{n_i}C'',$$

a map which on united K-theory agrees with the restriction of μ_1 to $K^{CRT}(S^{k_i}\mathbb{C})$ up to multiplication by β_T^{-1}.

We need one more homomorphism,

$$\beta_0 : S^{10}S^{-2}S^{\sim-8}\mathbb{R} \to S^{10}S^{-2}S^{\sim-8}(C^{\sim})$$

based on the unital inclusion $\mathbb{R} \to C^{\sim}$.

To assemble these homomorphisms, let $\mathcal{K}$ be the algebra of compact operators on a separable Hilbert space and let ϕ_i be a collection of mutually orthogonal inclusions from M_{n_i} to $\mathcal{K}$ for $i \in I_O \cup I_U \cup I_T \cup \{0\}$. Let

$$B' = S^{10}S^{-2}S^{\sim-8}\mathbb{R} \oplus \bigoplus_{i\in I_O} S^{10}S^{k_i-10}\mathbb{R} \oplus \bigoplus_{i\in I_U} S^{11}S^{k_i-9}\mathbb{C} \oplus \bigoplus_{i\in I_T} S^{11}S^{k_i-7}T$$

and we define $\beta : B' \to C' = \mathcal{K} \otimes C''$ by setting it to be $\phi_i \circ \beta_i$ on each summand.

Therefore, we have a geometric realization of μ_1 in the sense that β_* is injective and has the same cokernel as μ_1. Let A' be the mapping cone of β. Then we have a short exact sequence

$$0 \to SC' \to A' \to B' \to 0.$$

In the resulting long exact sequence, the homomorphism $K^{CRT}(B') \to K^{CRT}(SC')$ of degree -1 is the same as β_* (see [22, Proposition 2.5] or [10, Theorem 1.1]). Since β_* is injective, the long exact sequence collapses to the short exact sequence

$$0 \to K^{CRT}(B') \xrightarrow{\beta_*} K^{CRT}(C') \xrightarrow{i_*} K^{CRT}(A') \to 0,$$

where i_* has degree -1. The united K-theory of A' is thus a suspension of the CRT-module M; so the algebra $A = S^{-1}A'$ satisfies $K^{CRT}(A) \cong M$ as desired.

Since A is constructed from the commutative algebras $\mathbb{R}$, $\mathbb{C}$ and T using the operations of countable direct sum, suspensions, desuspensions, unitization, forming matrix algebras, stabilization, and forming mapping cones we know that A is separable, nuclear, and in the category of real C^*-algebras that satisfy the Universal Coefficient theorem. □

4. Unital

The goal of this section is to show that given a real C^*-algebra A, we can obtain a unital algebra with the same united K-theory. For this, we will use the real analog of the construction of [1, Proposition 4.1].

We begin by recording some results regarding real simple purely infinite C^*-algebras and their K-theory. These results are analogs of well-known results in the theory of complex simple purely infinite C^*-algebras. In each case, the proof follows directly from the corresponding result in the complex case, or can be proven in the same way as the complex version.

It is well known that the inclusion of a full corner in a complex C^*-algebra induces an isomorphism on K-theory. It is an easy consequence of Theorem 4 that the same is true for real C^*-algebras. For completeness, we record the proofs of both statements below.

Proposition 9.

(1) *Let p be a full projection in a complex C^*-algebra A. Then the inclusion $i : pAp \to A$ induces an isomorphism on K-theory.*
(2) *Let p be a full projection in a real C^*-algebra A. Then the inclusion $i : pAp \to A$ induces an isomorphism on united K-theory.*

Proof. Let A be a complex C^*-algebra and let p be a full projection. By [7, Lemma 2.5], there is a partial isometry $v \in M(pAp \otimes \mathcal{K})$ such that $v^*v = 1$ and $vv^* = p \otimes 1$. Replacing v by $(p \otimes 1)v$, we may assume that $v \in (pAp \otimes \mathcal{K})^+$. Then there is an isomorphism $\alpha : pAp \otimes \mathcal{K} \to A \otimes \mathcal{K}$ defined by $x \mapsto v^*xv$.

Now, if q is any projection in $(pAp \otimes \mathcal{K})^+$, then $(qv)^*(qv) = v^*qv$ and $(qv)(qv)^* = q(p \otimes 1)q^* = q$. Thus in $K_0(A^+)$ we have $[i(q)] = [q] = [v^*qv] = [\alpha(q)]$. Similarly, if u is any unitary in $(pAp \otimes \mathcal{K})^+$, then in $K_1(A^+)$ we have $[i(u)] = [u] = [v^*uv] = [\alpha(u)]$.

Therefore, i_* and α_* agree as homomorphisms from $K_*(pAp)$ to $K_*(A)$. Since α_* is an isomorphism, so is i_*. This proves part (1). To prove part (2), let p be a full projection in a real C^*-algebra A. By part (1) the inclusion i_* induces an isomorphism on complex K-theory $KU_*(pAp) \to KU_*(A)$. Therefore, i_* is an isomorphism on united K-theory by Theorem 4. □

Lemma 10. *Let p and q be non-trivial projections in a simple purely infinite C^*-algebra. Then there is a projection p' such that $p' \sim p$ and $p' < q$.*

The complex version of Lemma 10 can be found as [9, Proposition 1.5] or [11, Lemma V.5.4]. The proof of Lemma 10 follows exactly the proof of [11, Lemma V.5.4]. (This was also observed by Stacey in the proof of [24, Proposition 4.1].) Once this lemma is established, the proof of Proposition 11 follows exactly the proof of [9, Theorem 1.4].

Proposition 11. *Let A be a real simple purely infinite C^*-algebra. Then*

$$KO_0(A) \cong \{[p] \mid p \text{ is a non-zero projection in } A\},$$

where $[p]$ represents the Murray–von Neumann equivalence class of a projection p in A.

The next proposition is generalization of [1, Proposition 2.4.1], with a similar proof.

Proposition 12. *There is a functor F from the category of all real C^*-algebras (and real C^*-algebra homomorphisms) to the category of all real unital C^*-algebras (and real unital C^*-algebra homomorphisms) and a natural transformation $\eta : A \to F(A)$ which induces an isomorphism on united K-theory. Furthermore,*

(1) *If A is nuclear, then $F(A)$ is nuclear.*
(2) *If A is separable, then $F(A)$ is separable and η is a KK-equivalence.*
(3) *If A is separable and satisfies the UCT, then $F(A)$ satisfies the UCT.*

Proof. Let $\mathcal{O}_\infty^{\mathbb{R}}$ be the real Cuntz algebra generated by an infinite sequence of mutually orthogonal isometries. By Theorem 4 the unital inclusion $\mathbb{R} \to \mathcal{O}_\infty^{\mathbb{R}}$ induces an isomorphism on united K-theory since the complexification $\mathbb{C} \to \mathcal{O}_\infty$ induces an isomorphism on K-theory. By Proposition 11, there is a non-zero projection $e \in \mathcal{O}_\infty^{\mathbb{R}}$ and a projection $q < e$ such that $[e] = 0$ and $[q] = [1_{\mathcal{O}_\infty^{\mathbb{R}}}]$.

Since e is infinite, there exists a proper subprojection p_1 such that $p_1 \sim e$. Let $p_2 = e - p_1$. Then $[p_1] = [p_2] = [e] = 0$. Therefore (again by Proposition 11) there are partial isometries s_1 and s_2 in $e\mathcal{O}_\infty^{\mathbb{R}}e$ such that $s_i^* s_i = e$ and $s_i s_i^* = p_i$. Let $D = C^*(s_1, s_2)$. Then the algebra $D = C^*(s_1, s_2)$ is a unital subalgebra of $e\mathcal{O}_\infty^{\mathbb{R}}e$ which is isomorphic to $\mathcal{O}_2^{\mathbb{R}}$.

Now, for any real C^*-algebra A, let A^+ be the unitization of A and let $\pi_A : A^+ \to \mathbb{R}$ be the usual projection with kernel A. We define

$$F(A) = \{b \in e\mathcal{O}_\infty^{\mathbb{R}}e \otimes A^+ \mid (1 \otimes \pi_A)(b) \in D\}.$$

The element $e \otimes 1$ is a unit for $F(A)$. The natural transformation $\eta : A \to F(A)$ is defined by $a \mapsto q \otimes a$.

We will show that η induces an isomorphism on united K-theory. Note that η is a composition of the homomorphism $A \to e\mathcal{O}_\infty^{\mathbb{R}}e \otimes A$ defined by $a \mapsto q \otimes a$ and the inclusion $e\mathcal{O}_\infty^{\mathbb{R}}e \otimes A \hookrightarrow F(A)$. The homomorphism $A \to e\mathcal{O}_\infty^{\mathbb{R}}e \otimes A$ induces an isomorphism on united K-theory because the map $\mathbb{R} \to e\mathcal{O}_\infty^{\mathbb{R}}e$ defined by $t \mapsto tq$ does using the Künneth formula for united K-theory [2]. Secondly, the inclusion $e\mathcal{O}_\infty^{\mathbb{R}}e \otimes A \hookrightarrow F(A)$ induces an isomorphism on united K-theory because of the short exact sequence

$$0 \to e\mathcal{O}_\infty^{\mathbb{R}}e \otimes A \hookrightarrow F(A) \xrightarrow{1\otimes\pi_A} D \to 0$$

and the fact that $K^{CRT}(D) = K^{CRT}(\mathcal{O}_2) = 0$. It follows that η induces an isomorphism on united K-theory.

From the short exact sequence above, if A is separable or nuclear then the same is true of $F(A)$. Furthermore, the argument of the previous paragraph also works for KK-theory, showing that η induces isomorphisms

$$KK^{CRT}(B, A) \to KK^{CRT}\big(B, F(A)\big)$$

and

$$KK^{CRT}\big(F(A), B\big) \to KK^{CRT}(A, B)$$

for any real separable C^*-algebra B. If A is separable, then so is $F(A)$ and by the Yoneda lemma, η induces a KK-equivalence. In particular, if A is separable and satisfies the UCT, so does $F(A)$. $\square$

5. Simple and purely infinite

In [14] Kumjian presents a construction (based on a special case of Pimsner's construction in [18]) which turns any complex separable unital C^*-algebra A into a complex C^*-algebra $\mathcal{O}_E$ which is simple and purely infinite such that there is an inclusion $A \hookrightarrow \mathcal{O}_E$ which is a (complex) KK-equivalence. In this section, we show that this construction can be carried out in the real case. Combined with the results from Sections 3 and 4, this will complete the proof of Theorem 1.

Proposition 13. *Let A be a real separable unital C^*-algebra. Then there is a real separable simple purely infinite C^*-algebra $\mathcal{O}_E^{\mathbb{R}}$ and a unital inclusion $A \hookrightarrow \mathcal{O}_E^{\mathbb{R}}$ which induces an isomorphism on united K-theory. Furthermore, if A is nuclear and satisfies the UCT, then the same is true of $\mathcal{O}_E^{\mathbb{R}}$ and ι is a* (*real*) *KK-equivalence.*

Recall that a complex C^*-algebra A is said to have a real structure if there is a conjugate linear involution $x \mapsto \bar{x}$. In that case, the set $A^{\mathbb{R}}$ of fixed points is a real C^*-algebra. Conversely, given a real C^*-algebra A, the complexification $A_{\mathbb{C}}$ has a real structure given by $a_1 + ia_2 \mapsto a_1 - ia_2$. These functors are inverse to each other so there is a bijection between complex C^*-algebras with real structure and real C^*-algebras. To prove Proposition 13 we will retrace Kumjian's construction, showing that the real structure of $A_{\mathbb{C}}$ passes to $\mathcal{O}_E$.

Definition 14. Let A be a complex C^*-algebra with a real structure.

(1) A Hilbert A-module E is said to have a real structure if E has a conjugate linear involution $e \mapsto \bar{e}$ that satisfies $\overline{\langle e, f\rangle} = \langle \bar{e}, \bar{f}\rangle$ and $\overline{e \cdot a} = \bar{e} \cdot \bar{a}$ for all $a \in A$ and $e, f \in E$.
(2) A Hilbert A-bimodule (E, ϕ) is said to have a real structure if the Hilbert A-module E has a real structure as in part (1) and the homomorphism $\phi : A \to \mathcal{L}(E)$ satisfies $\overline{\phi(a)e} = \phi(\bar{a})\bar{e}$ for all $a \in A$ and $e \in E$.

If E is a Hilbert A-module with a real structure, then the C^*-algebra $\mathcal{L}(E)$ has a real structure defined by $\overline{T}(e) = \overline{T(\bar{e})}$ for all $T \in \mathcal{L}(E)$ and $e \in E$. With this language, the Hilbert bimodule condition above can be restated as $\overline{\phi(a)} = \phi(\bar{a})$, interpreted as saying that the $*$-homomorphism $\phi : A \to \mathcal{L}(E)$ respects the real structures.

Let A be a C^*-algebra and let H be a Hilbert space, both with a real structures and let $\pi : A \to \mathcal{L}(H)$ be a representation which respects the real structures. For example, this can be obtained by complexifying any representation of a real C^*-algebra on a real Hilbert. Also assume that $\pi(A) \cap \mathcal{K}(H) = \{0\}$. Following Kumjian, we define a Hilbert A-bimodule (E, ϕ) by

$$E = H \otimes_{\mathbb{C}} A$$

with bimodule structure given by $(\xi \otimes a) \cdot b = \xi \otimes (a \cdot b)$ and $\phi(b)(\xi \otimes a) = \pi(b)\xi \otimes a$ for all $a, b \in A$ and $\xi \in H$. We give (E, ϕ) a real structure by $\overline{\xi \otimes a} = \bar{\xi} \otimes \bar{a}$.

Similarly, the Fock space

$$\mathcal{E}_+ = \bigoplus_{n=0}^{\infty} E^{\otimes n}$$

is also a Hilbert A-bimodule with a real structure. The involution is defined on pure tensors by

$$\overline{e_1 \otimes e_2 \otimes \cdots \otimes e_n} = \overline{e_1} \otimes \overline{e_2} \otimes \cdots \otimes \overline{e_n}.$$

For any element $e \in E$, we define the operator $T_e \in \mathcal{L}(\mathcal{E}_+)$ on pure tensors by $T_e(e_1 \otimes \cdots \otimes e_n) = e \otimes e_1 \otimes \cdots \otimes e_n$. Since $\overline{T_e} = T_{\bar{e}}$, the involution of $\mathcal{L}(E)$ restricts to an involution of the algebra $\mathcal{T}_E$ generated by $\{T_e\}_{e \in E}$.

In the general case, $\mathcal{O}_E$ is the quotient of $\mathcal{T}_E$ by the C^*-algebra generated in $\mathcal{L}(\mathcal{E}_+)$ by $\mathcal{L}(\bigoplus_{n=0}^{N} E^{\otimes n})$ for all positive integers N. But under the assumption $\pi(A) \cap \mathcal{K}(H) = \{0\}$, we have $\mathcal{O}_E \cong \mathcal{T}_E$ (see [18, Corollary 3.14]). In either case, the involution of $\mathcal{L}(\mathcal{E}_+)$ induces one on $\mathcal{O}_E$. Furthermore, the inclusion $\iota : A \hookrightarrow \mathcal{O}_E$ given by $\iota(a)(e_1 \otimes e_2 \otimes \cdots \otimes e_n) = \phi(a)(e_1) \otimes e_2 \otimes \cdots \otimes e_n$, respects the real structures of A and $\mathcal{O}_E$.

If we begin with a real separable unital C^*-algebra A, then the complexification $A_{\mathbb{C}}$ has a real structure and the construction above yields an inclusion $\iota : A \to \mathcal{O}_E^{\mathbb{R}}$, where $\mathcal{O}_E^{\mathbb{R}}$ is the fixed point set of $\mathcal{O}_E$.

Proof of Proposition 13. Let A be a real separable unital C^*-algebra. Applying the construction above, we obtain an inclusion $\iota : A \to \mathcal{O}_E^{\mathbb{R}}$. By [14, Theorem 2.8], $\mathcal{O}_E$ is simple and purely infinite. Thus $\mathcal{O}_E^{\mathbb{R}}$ is simple and purely infinite by [24, Theorem 3.3]. By [18, Corollary 4.5], the inclusion $\iota : A_{\mathbb{C}} \to \mathcal{O}_E$ is a KK-equivalence. In particular, it induces an isomorphism on K-theory, so by Theorem 4, $\iota : A \to \mathcal{O}_E^{\mathbb{R}}$ induces an isomorphism on united K-theory.

If A is nuclear and satisfies the UCT, then by [14, Theorem 3.1], the same is true of $\mathcal{O}_E$. Thus $\mathcal{O}_E^{\mathbb{R}}$ is nuclear and satisfies the UCT. In particular, since A and $\mathcal{O}_E^{\mathbb{R}}$ have isomorphic united K-theory and both satisfy the UCT, they are KK-equivalent. □

The proof of [18, Section 4] will probably carry over to show that ι is a (real) KK-equivalence in general, giving a stronger statement than our Theorem 13, but we do not need this for our present purposes.

Note that absent a full classification theorem for real simple purely infinite C^*-algebras, there is no guarantee that $\mathcal{O}_E^{\mathbb{R}}$ is independent of the choice of π (as $\mathcal{O}_E$ is when $A_{\mathbb{C}}$ is nuclear and satisfies the UCT).

Proof of Theorem 1. Let M be a countable acyclic CRT-module. By Theorem 5, there is a real separable nuclear C^*-algebra A_1 satisfying the UCT such that $K^{CRT}(A_1) \cong M$. Applying the functor F of Proposition 12, there is a real separable nuclear unital C^*-algebra A_2 satisfying the UCT such that $K^{CRT}(A_2) \cong M$. Then applying the real Kumjian construction (Proposition 13), there is a real separable nuclear unital simple purely infinite A_3 satisfying the UCT such that $K^{CRT}(A_3) \cong M$. Finally, let $A_4 = \mathcal{K}(H) \otimes_{\mathbb{R}} A_3$ where H is a real separable Hilbert space. By Lemma 15, A_4 is purely infinite and is the real C^*-algebra needed to prove part (1).

Now, let m be any element in M_0^O. By Proposition 11 there is a projection $p \in A_4$ such that $[p] = m$. Let A_5 be the corner algebra pA_4p. By Proposition 9, $K^{CRT}(A_5) \cong M$. This proves part (2). □

Lemma 15. *If A is a real purely infinite simple C^*-algebra, then the stabilization $\mathcal{K}(H) \otimes_{\mathbb{R}} A$ is also purely infinite and simple.*

Proof. By [24, Lemma 4.2], the matrix algebras $M_n(A)$ are purely infinite. The proof of [19, Proposition 4.1.8] carries over immediately to the real case to show that the inductive limit of simple purely infinite C^*-algebras is again simple and purely infinite. □

6. Applications

The following result shows that K-theory by itself cannot classify isomorphism classes or even KK-equivalence of real simple purely infinite C^*-algebras.

Theorem 16. *There exist two real Kirchberg algebras A and B such that $KO_*(A) \cong KO_*(B)$, but $K^{CRT}(A) \ncong K^{CRT}(B)$.*

Proof. By Theorem 1, it suffices to find two distinct countable acyclic CRT-modules whose real parts are isomorphic. I am indebted to A.K. Bousfield for sharing with me the example of such CRT-modules.

Let (G, α) be a group with involution satisfying $\ker(1+\alpha) = \operatorname{image}(1-\alpha)$ and $\ker(1-\alpha) = \operatorname{image}(1+\alpha)$. For the groups $G^+ = \{g \in G \mid \alpha(g) = g\}$ and $G^- = \{g \in G \mid \alpha(g) = -g\}$ there are exact sequences

$$0 \to G^+ \xrightarrow{i^+} G \xrightarrow{\pi^-} G^- \to 0$$

Table 1
$P(G,\alpha)$

n	0	1	2	3	4	5	6	7	8
$N(G)^O$	G^+	0	G^-	0	G^+	0	G^-	0	G^+
$N(G)^U$	G	0	G	0	G	0	G	0	G
$N(G)^T$	G^+	G^-	G^-	G^+	G^+	G^-	G^-	G^+	G^+
$(\eta_O)_n$	0	0	0	0	0	0	0	0	0
c_n	i^+	0	i^-	0	i^+	0	i^-	0	i^+
r_n	π^+	0	π^-	0	π^+	0	π^-	0	π^+
ε_n	1	0	1	0	1	0	1	0	1
ζ_n	i^+	0	i^-	0	i^+	0	i^-	0	i^+
$(\psi_U)_n$	α	0	$-\alpha$	0	α	0	$-\alpha$	0	α
$(\psi_T)_n$	1	-1	1	-1	1	-1	1	-1	1
γ_n	π^+	0	π^-	0	π^+	0	π^-	0	π^+
τ_n	0	1	0	1	0	1	0	1	0

and

$$0 \to G^- \xrightarrow{i^-} G \xrightarrow{\pi^+} G^+ \to 0,$$

where $\pi^+ = 1+\alpha$, $\pi^- = 1-\alpha$, and i^+ and i^- are inclusion homomorphisms.

Then it can be easily verified that the groups and natural transformations in Table 1 form an acyclic *CRT*-module $P(G,\alpha)$.

Let $G = \mathbb{Z}_2^4$ and $H = \mathbb{Z}_4 \oplus \mathbb{Z}_2^2$ with involutions

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. For both groups with involution, we have $\operatorname{image}\pi^+ = \ker\pi^-$, $\operatorname{image}\pi^- = \ker\pi^+$. Furthermore, $G^+ \cong H^+ \cong \mathbb{Z}_2^2$, and $G^- \cong H^- \cong \mathbb{Z}_2^2$. Thus the real parts of $P(G,\alpha)$ and $P(H,\beta)$ agree, even taking into account the actions of η_O and ξ, while the complex parts do not. □

Using another *CRT*-module construction, we can prove that every complex Kirchberg algebra satisfying the UCT is the complexification of a real C^*-algebra. In fact, there is at least one different real structure for each involution on K-theory.

Theorem 17. *Let A be any complex Kirchberg algebra satisfying the UCT and let α be a graded involution of $K_*(A)$. Then A is isomorphic to the complexification of a real C^*-algebra $A_{\mathbb{R}}$ in such a way that $\alpha = \psi_U : KU_*(A_{\mathbb{R}}) \to KU_*(A_{\mathbb{R}})$.*

Since real structures of complex C^*-algebras correspond with anti-multiplicative involutions we have the following corollary.

Corollary 18. *Let A be any complex Kirchberg algebra satisfying the UCT and let α be a graded involution of $K_*(A)$. Then there is an anti-multiplicative involution of Ψ of A such that $\Psi_* = \alpha$.*

Proof. If A is the complexification of a real C^*-algebra, then the anti-multiplicative involution Ψ on A is the composition of the conjugation (which induces ψ_U on K-theory) and the adjoint (which induces the identity). $\square$

For the proof of Theorem 17, we need the following lemma.

Lemma 19 *(Hewitt). Let G be a group with involution α. Then there exists a CRT-module $N(G,\alpha)$ such that $N(G,\alpha)^U_0 = G$ and $N(G,\alpha)^U_1 = 0$ and $(\psi_U)_0 = \alpha$.*

The construction will be made in terms of the core of a *CRT*-module as defined by Beatrice Hewitt and briefly introduced here. In Section 5.1 of her dissertation [12], Hewitt showed that acyclic *CRT*-modules can be classified in terms of their cores, which contains only the complex part and the image of η_O in the real part (and some natural transformations). Thus the self-conjugate part of united K-theory is strictly unnecessary. We know of no way to express *CRT* tensor product or Hom functors in terms of just the cores, so for purposes of the Künneth formula and the universal coefficient theorem, it is still necessary to work with the full united K-theory. However, Hewitt's work does imply that to specify an acyclic *CRT*-module, it is enough to specify the core of the *CRT*-module.

Let M be an acyclic *CRT*-module. We define a derivation d on M_U by $d = (1+\psi_U)\beta_U^{-1} = \beta_U^{-1}(1-\psi_U)$ and then define

$$h_*M^U = \frac{\ker d}{\operatorname{image} d} = \frac{\ker(\beta_U^{-1}(1-\psi_U))}{\operatorname{image}((1+\psi_U)\beta_U^{-1})} = \frac{\ker(1-\psi_U)}{\operatorname{image}(1+\psi_U)}.$$

Let $\eta_O M^O$ denote the image of η_O in M^O. There is a homomorphism $c' : \eta_O M^O \to h_*M^U$ of degree -1 defined by $c'(\eta_O x) = [c(x)]$. We show that this is well defined. The *CRT*-module relation $(1-\psi_U)c(x) = 0$ implies that $c(x)$ is a cycle of h_*M^U. If $\eta_O x = 0$ then the acyclicity of M implies $x = r(y)$ for some $y \in M^U$ and thus $c'(\eta_O x) = [cr(y)] = [(1+\psi_U)(y)] = 0$. There is also a homomorphism $r' : h_*M^U \to \eta_O M^O$ of degree -2 defined by $r'[y] = r\beta_U^{-1}y$. If $y \in \operatorname{image}(1+\psi_U)$ then $r'[y] = 0$ since $r\beta_U^{-1}(1+\psi_U) = r(1-\psi_U)\beta_U^{-1} = 0$. For $y \in \ker(1-\psi_U)$ we have $cr\beta_U y = (1-\psi_U)\beta_U y = \beta_U(1-\psi_U)y = 0$ which implies $r\beta_U y \in \operatorname{image}\eta_O$.

Because of the exact sequence

$$\cdots \to M^O_n \xrightarrow{\eta_O} M^O_{n+1} \xrightarrow{c} M^U_{n+1} \xrightarrow{r\beta_U^{-1}} M^O_{n-1} \to \cdots$$

it can be shown that

$$\cdots \to \eta_O M^O_n \xrightarrow{\eta_O} \eta_O M^O_{n+1} \xrightarrow{c'} h_*M^U_{n-1} \xrightarrow{r'} \eta_O M^O_n \to \cdots$$

is also an exact sequence.

Given an acyclic *CRT*-module M, the core of M, denoted core(M), consists of two graded groups$\{\eta_O M^O, M^U\}$ together with the homomorphisms $\{\beta_U, \beta_O, \eta_O, c', r', \psi_U\}$.

The core functor takes values in the category of abstract cores. An abstract core [12, Definition 6.0.1] is defined to be any pair of graded abelian groups $\{D_*, C_*\}$ such that D_* is a graded $\mathbb{Z}_2$-module, together with homomorphisms

$$\beta_U : C_* \to C_{*+2}, \qquad \beta_O : D_* \to D_{*+8}, \qquad \eta_O : D_* \to D_{*+1},$$
$$c' : \eta_O D_* \to h_{*-1} C, \qquad r' : h_* C \to \eta_O D_{*-2}, \qquad \psi_U : C_* \to C_*$$

such that β_U, ψ_U, and β_O are isomorphisms, the relations

$$\begin{aligned} \psi_U^2 &= 1, & \psi_U \beta_U &= -\beta_U \psi_U, \\ \eta_O^2 &= 0, & \eta_O \beta_O &= \beta_O \eta_O, \\ \beta_O r' &= r' \beta_U^4, & c' \beta_O &= \beta_U^4 c', \\ r' \beta_U^2 c' &= \eta_O, & \beta_U^2 c' r' &= c' r' \beta_U^2 \end{aligned}$$

hold, and the sequence

$$\cdots \to D_* \xrightarrow{\eta_O} D_{*+1} \xrightarrow{c'} h_* C \xrightarrow{r'} D_{*-2} \to \cdots$$

is exact.

The theorem below summarizes results from [12] which show that the core functor gives an equivalence between the category of isomorphism classes of *CRT*-modules with *CRT*-module homomorphisms and the category of isomorphism classes of abstract cores with homomorphisms of cores.

Theorem 20. [12, Theorems 7.3.1 and 7.3.3]

(1) *For each abstract core $\{D_*, C_*\}$, there is an exact CRT-module M such that the* $\mathrm{core}(M) \cong \{D_*, C_*\}$.
(2) *Let M and M' be exact CRT-modules. Each map $f : \mathrm{core}(M) \to \mathrm{core}(M')$ is induced by a map $\overline{f} : M \to M'$.*

Proof of Lemma 19. This construction is from [12, Section 8.4].

Since $\alpha^2 = 0$, we have $(1+\alpha)(1-\alpha) = (1-\alpha)(1+\alpha) = 0$. Let

$$G' = \frac{\ker(1-\alpha)}{\mathrm{image}(1+\alpha)} \quad \text{and} \quad G'' = \frac{\ker(1+\alpha)}{\mathrm{image}(1-\alpha)}.$$

Then Table 2 shows an abstract core $\{D_*, C_*\}$. Using [12, Theorem 7.3.1], let $N(G,\alpha)$ be the

Table 2
$\mathrm{core}(N(G,\alpha))$

n	0	1	2	3	4	5	6	7	8
D_*	0	G'	G'	G''	G''	0	0	0	0
C_*	G	0	G	0	G	0	G	0	G
$h_*(C)$	G'	0	G''	0	G'	0	G''	0	G'
$(\eta_O)_n$	0	1	0	1	0	0	0	0	0
c'_n	0	1	0	1	0	0	0	0	0
r'_n	0	0	0	0	1	0	1	0	0
$(\psi_U)_n$	α	0	$-\alpha$	0	α	0	$-\alpha$	0	α

unique acyclic CRT-module whose core is $\{D_*, C_*\}$. Then by construction we have

$$N(G,\alpha)_0^U = G, \qquad N(G,\alpha)_1^U = 0, \quad \text{and} \quad (\psi_U)_0 = \alpha. \qquad \square$$

Proof of Theorem 17. By Theorem 1 it suffices to find a CRT-module M such that

$$M^U = K_*(A) \quad \text{and} \quad \psi_U = \alpha.$$

Using Lemma 19, take $M = N(K_0(A), \alpha_0) \oplus \Sigma N(K_1(A), \alpha_1)$. $\square$

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